

# A simple argument for dark matter as an effect of slightly modified gravity

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## Abstract

This note presents a simple argument showing that dark matter is an effect of  $f(R)$  gravity based on the definition of slightly modified gravitational theories previously proposed by the author.

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## 1 Introduction

Discussions of dark matter and dark energy have led to the hypothesis that Einstein's gravitational theory does not hold on very large scales. An example of such a modification is the replacement of the Ricci scalar  $R$  by  $f(R)$  in the Einstein-Hilbert action to yield

$$S_{f(R)} = \int \sqrt{-g} f(R) d^4x. \quad (1)$$

Apart from the gravitational effects, no direct evidence of dark matter has ever been found. That dark matter is a geometric effect of  $f(R)$  gravity has already been shown by Böhmer, Harko, and Lobo [1]. All that is required is a small change in the Ricci scalar. The purpose of this note is to present a simple alternative argument based on a slight modification of  $f(R)$  gravity as defined by Kuhfittig [2].

## 2 Galactic rotation curves

An important objective in any modified gravitational theory is to explain the peculiar behavior of galactic rotation curves without postulating the existence of dark matter: test particles move with constant tangential velocity  $v_{\text{tg}}$  in a circular path. It is noted in

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Ref. [1] that galactic rotation curves generally show much more complicated dynamics. For present purposes, however, the analysis can be restricted to the region in which the velocity is indeed constant. In this note we will go a step further and confine the analysis to a narrow band around a constant  $r = a$ , the size of which is to be specified later. A constant radius is of particular interest since for the motion of particles in circular orbits, the potential  $V(r)$  must satisfy certain conditions if the orbit is to be stable.

To meet the limited goals in this note, we need only a few basic facts about galactic rotation curves [4, 5, 6, 7, 8]. We start with the line element

$$ds^2 = -e^{\phi(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where  $e^{\phi(r)} = B_0 r^l$  and  $l = 2v_{\text{tg}}^2 \approx 0.000001$ . (We are using units in which  $c = G = 1$ .) According to Ref. [5], for a test particle with four-velocity  $U^\alpha = dx^\alpha/d\tau$  we have

$$\dot{r}^2 = E^2 + V(r), \quad (3)$$

where  $\dot{r} = dr/d\tau$  and the potential is

$$V(r) = -E^2 + E^2 \frac{e^{-\lambda}}{B_0 r^l} - e^{-\lambda} \left( 1 + \frac{L^2}{r^2} \right). \quad (4)$$

Moreover, if the circular orbits are given by  $r = a$ , then

$$\frac{da}{d\tau} = 0 \quad \text{and} \quad \left. \frac{dV}{dr} \right|_{r=a} = 0.$$

Also, two conserved quantities, the angular momentum  $L$  and the relativistic energy  $E$ , are given by [6, 7]

$$L = \pm \sqrt{\frac{l}{2-l}} a \quad (5)$$

and

$$E = \pm \sqrt{\frac{2B_0}{2-l}} a^{l/2}. \quad (6)$$

The orbits are stable if

$$\left. \frac{d^2V}{dr^2} \right|_{r=a} < 0. \quad (7)$$

### 3 Slightly modified $f(R)$ gravity

As noted earlier, the radius  $r = a$  will be specified later. For now it is sufficient to recall that we will stay in a narrow band around  $r = a$ . The importance of this assumption lies in the fact that, due to its simple form,  $e^{\phi(r)} = B_0 r^l \approx \text{constant}$ , so we may assume that  $\phi' \equiv 0$ . (By contrast, the behavior of  $e^{\lambda(r)}$  is much more volatile in the present situation, as we will see.) The assumption  $\phi' \equiv 0$  allows us to define the concept of “slightly modified gravity,” as discussed by Kuhfittig [2]. To that end, we list the gravitational field equations in the form used by Lobo and Oliveira [3]:

$$\rho(r) = F(r) \frac{b'(r)}{r^2}, \quad (8)$$

$$p_r(r) = -F(r)\frac{b(r)}{r^3} + F'(r)\frac{rb'(r) - b(r)}{2r^2} - F''(r)\left[1 - \frac{b(r)}{r}\right], \quad (9)$$

$$p_t(r) = -\frac{F'(r)}{r}\left[1 - \frac{b(r)}{r}\right] + \frac{F(r)}{2r^3}[b(r) - rb'(r)], \quad (10)$$

where  $F = \frac{df}{dR}$ . The curvature scalar is given by

$$R(r) = \frac{2b'(r)}{r^2}. \quad (11)$$

Observe that Eqs. (8)-(10) reduce to the Einstein field equations for  $\phi' \equiv 0$  whenever  $F \equiv 1$ . So comparing Eqs. (8) and (11), a slight change in  $F$  results in a slight change in  $R$ . According to Eq. (1), this characterizes  $f(R)$  modified gravity. This change can be quantified by assuming that  $F(r)$  remains close to unity and relatively flat, i.e.,  $|F'(r)|$  is relatively close to zero.

## 4 The solution

Our first step is to expand  $F(r)$  in a Taylor series around  $r = a$ :

$$F(r) = F(a) + F'(a)(r - a) + \frac{1}{2}F''(a)(r - a)^2 + \dots$$

Since  $r$  is assumed to be close to  $r = a$ , higher-order terms become negligible. So

$$F(r) = F(a) + F'(a)(r - a)$$

and  $F(r)|_{r=a} = F(a) \approx 1$ , while  $F'(a) \approx 0$ . Observe that  $F''$  can be omitted. Even though  $r = a$  has not been specified, we can assume that  $F(a)$  and  $F'(a)$  can be treated as parameters in Eqs. (8)-(10) since the precise values are not needed.

Next, we take the equation of state to be

$$p = m\rho, \quad 0 < m < 1,$$

describing normal matter and also implying that  $p_r = p_t$ , since a cosmological setting assumes a homogeneous distribution of matter. It now follows from Eqs. (8) and (9) that

$$mF\frac{b'}{r^2} = -F\frac{b}{r^3} + F'\frac{rb' - b}{2r^2}.$$

After some rearrangement, we get

$$\frac{b'}{b} = \frac{2F + rF'}{r(rF' - 2mF)} = -\frac{1/m}{r} + \frac{2(1+m)F}{rF' - 2mF}.$$

Integrating, we have (since  $F$  and  $F'$  are assumed to be constants)

$$\ln b(r) = -\frac{1}{m}\ln r + 2(1+m)F\frac{1}{F'}\ln|rF' - 2mF| + \ln c, \quad c > 0,$$

or

$$b(r) = cr^{-1/m}|rF' - 2mF|^{2(1+m)F/F'}, \quad c > 0. \quad (12)$$

Thus

$$e^{-\lambda(r)} = 1 - \frac{b(r)}{r} = 1 - cr^{-1/m-1}|rF' - 2mF|^{2(1+m)F/F'}, \quad c > 0. \quad (13)$$

Before returning to the stability question, we need to show that thanks to the properties of  $F(r)$ ,  $e^{-\lambda(r)} \approx \text{constant}$ . First recall that  $|F'(a)|$  is close to zero for any  $a$ . So  $e^{-\lambda(r)}$  in Eq. (13) could get large beyond any bound as  $F'(a) \rightarrow 0$ . (At this point,  $r = a$  is still unspecified.) We conclude that in order to get a meaningful result,  $|rF'(a) - 2mF(a)|$  has to be less than unity whenever  $F'(a) > 0$ , thereby causing  $b(r)/r$  to become negligible for  $F'(a)$  sufficiently close to zero. Since  $r$  is assumed to be close to  $a$ , by letting  $r \approx a$ , we get the following approximate inequality:

$$|aF'(a) - 2mF(a)| \lesssim 1, \quad (r \approx a),$$

or

$$-1 \lesssim aF'(a) - 2mF(a) \lesssim 1.$$

Solving,

$$\frac{2mF(a) - 1}{F'(a)} \lesssim a \lesssim \frac{2mF(a) + 1}{F'(a)}, \quad (14)$$

which is the approximate range of  $a$  for the case  $F'(a) > 0$ . For example, if  $m \lesssim 1$  and  $F \approx 1$ , then  $1/F'(a) \lesssim a \lesssim 3/F'(a)$ . The interval widens to a large region as  $m$  gets closer to  $1/2$ . If  $F'(a) < 0$ , then we must have

$$|aF'(a) - 2mF(a)| \gtrsim 1,$$

leading to

$$a \lesssim \frac{2mF(a) + 1}{F'(a)} \quad \text{or} \quad a \gtrsim \frac{2mF(a) - 1}{F'(a)}, \quad (15)$$

where the left inequality is disregarded since  $a$  is positive. We now see that the previously unspecified radius  $a$  actually satisfies the approximate inequalities (14) and (15), as a result of which  $e^{-\lambda(r)} \approx \text{constant}$ . (We will also use these inequalities in the next section to show that we do not obtain any stable orbits in Einstein gravity.)

Returning now to Eq. (4), we can write

$$V(r) = -E^2 + e^{-\lambda} \left[ \frac{2}{2-l} \left( \frac{a}{r} \right)^l - 1 - \frac{l}{2-l} \left( \frac{a}{r} \right)^2 \right] \quad (16)$$

and define

$$G(r) = \frac{2}{2-l} \left( \frac{a}{r} \right)^l - 1 - \frac{l}{2-l} \left( \frac{a}{r} \right)^2.$$

Observe that  $G(a) = 0$  and

$$G'(r)|_{r=a} = -\frac{a}{r^2} \left[ \frac{2l}{2-l} \left( \frac{a}{r} \right)^{l-1} - \frac{2l}{2-l} \left( \frac{a}{r} \right) \right] \Big|_{r=a} = 0.$$

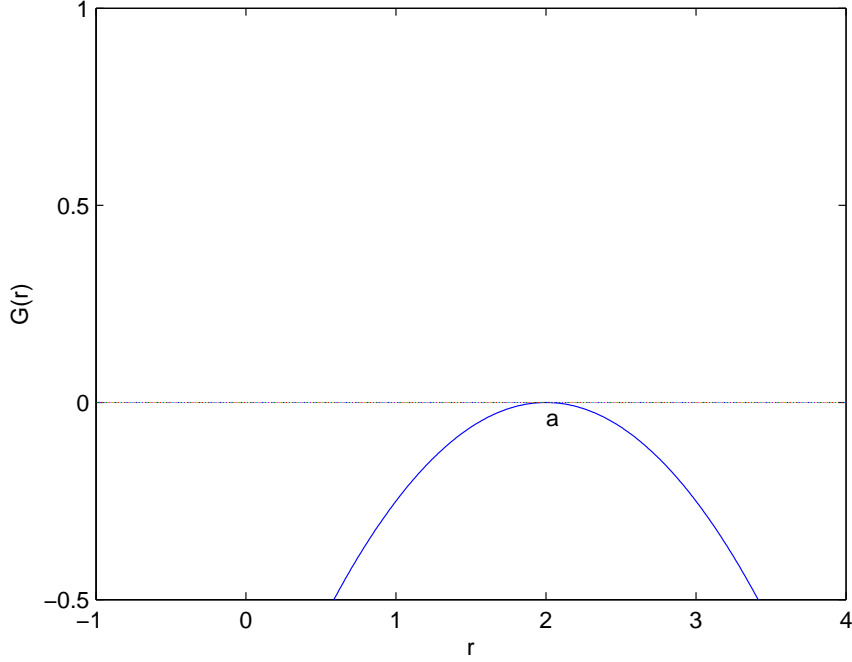


Figure 1:  $G(r) < 0$  near  $r = a$ .

Continuing with these calculations, we find that

$$G''(a) = -\frac{2l}{a^2} < 0.$$

So  $G(r) < 0$  in the neighborhood of  $r = a$ . [See Fig. 1.]

We now have  $\dot{a} = 0$  and  $V'(a) = 0$ . Since  $-E^2$  and  $e^{-\lambda}$  are constants, it follows directly from Eq. (16) that

$$V''(a) < 0, \tag{17}$$

thereby showing that we have stable orbits.

## 5 Comparison to Einstein gravity

In this section we examine the limiting case  $F \rightarrow 1$ , corresponding to Einstein gravity. Suppose in the equation of state  $p = m\rho$  we have  $m \gtrsim \frac{1}{2}$ . Then inequality (14) yields

$$a \gtrsim \frac{2mF(a) - 1}{F'(a)} > 0.$$

But the same relationship is obtained from inequality (15) whenever  $m \lesssim \frac{1}{2}$ . Now, as we get close to Einstein gravity,  $F'(a)$  gets close to zero regardless of the value of  $a$ . So in either case,  $a \rightarrow \infty$  as  $F' \rightarrow 0$  and we do not get a (finite) stable orbit in Einstein gravity.

## 6 Conclusion

This note gives a simple argument showing that stable galactic orbits are an effect of  $f(R)$  gravity without dark matter by assuming a particular definition of slightly modified gravity [2]: the function  $F(r)$  in the Einstein field equations is characterized by the properties  $F(r) \approx 1$  and  $|F'(r)| \approx 0$ . Given the equation of state  $p = m\rho$ ,  $0 < m < 1$ , describing normal matter, we obtain stable circular orbits, namely  $r = a$ , satisfying the following approximate inequalities: if  $F'(a) > 0$ , then

$$\frac{2mF(a) - 1}{F'(a)} \lesssim a \lesssim \frac{2mF(a) + 1}{F'(a)};$$

if  $F'(a) < 0$ , then

$$a \gtrsim \frac{2mF(a) - 1}{F'(a)}.$$

These inequalities indicate that such stable orbits exist in a large region. There are no stable orbits in the limiting case,  $F \rightarrow 1$ ,  $F' \rightarrow 0$ , corresponding to Einstein gravity, without hypothesizing the existence of dark matter.

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